Archimedes of Syracuse

Born: 287 BC in Syracuse, Sicily
Died: 212 BC in Syracuse, Sicily

Archimedes' father was Phidias, an astronomer. We know nothing else about Phidias other than this one fact and we only know this since Archimedes gives us this information in one of his works, *The Sandreckoner.* A friend of Archimedes called Heracleides wrote a biography of him but sadly this work is lost. How our knowledge of Archimedes would be transformed if this lost work were ever found, or even extracts found in the writing of others.

Archimedes was a native of Syracuse, Sicily. It is reported by some authors that he visited Egypt and there invented a device now known as Archimedes' screw. This is a pump, still used in many parts of the world. It is highly likely that, when he was a young man, Archimedes studied with the successors of Euclid in Alexandria. Certainly he was completely familiar with the mathematics developed there, but what makes this conjecture much more certain, he knew personally the mathematicians working there and he sent his results to Alexandria with personal messages. He regarded Conon of Samos, one of the mathematicians at Alexandria, both very highly for his abilities as a mathematician and he also regarded him as a close friend.

In the preface to *On spirals* Archimedes relates an amusing story regarding his friends in Alexandria. He tells us that he was in the habit of sending them statements of his latest theorems, but without giving proofs. Apparently some of the mathematicians there had claimed the results as their own so Archimedes says that on the last occasion when he sent them theorems he included two which were false [3]:-

... so that those who claim to discover everything, but produce no proofs of the same, may be confuted as having pretended to discover the impossible.

Other than in the prefaces to his works, information about Archimedes comes to us from a number of sources such as in stories from Plutarch, Livy, and others. Plutarch tells us that Archimedes was related to King Hieron II of Syracuse (see for example [3]):-

Archimedes ... in writing to King Hiero, whose friend and near relation he was....

Again evidence of at least his friendship with the family of King Hieron II comes from the fact that *The Sandreckoner* was dedicated to Gelon, the son of King Hieron.

There are, in fact, quite a number of references to Archimedes in the writings of the time for he had gained a reputation in his own time which few other mathematicians of this period achieved. The reason for this was not a widespread interest in new mathematical ideas but rather that Archimedes had invented many machines which were used as engines of war. These were particularly effective in the defence of Syracuse when it was attacked by the Romans under the command of Marcellus.

Plutarch writes in his work on Marcellus, the Roman commander, about how Archimedes' engines of war were used against the Romans in the siege of 212 BC:-

... when Archimedes began to ply his engines, he at once shot against the land forces all sorts of missile weapons, and immense masses of stone that came down with incredible noise and violence; against which no man could stand; for they knocked down those upon whom they fell in heaps, breaking all their ranks and files. In the meantime huge poles thrust out from the walls over the ships and sunk some by great weights which they

let down from on high upon them; others they lifted up into the air by an iron hand or beak like a crane's beak and, when they had drawn them up by the prow, and set them on end upon the poop, they plunged them to the bottom of the sea; or else the ships, drawn by engines within, and whirled about, were dashed against steep rocks that stood jutting out under the walls, with great destruction of the soldiers that were aboard them. A ship was frequently lifted up to a great height in the air (*a dreadful thing to behold*), *and was rolled to and fro, and kept swinging, until the mariners were all thrown out, when at length it was dashed against the rocks, or let fall.*

Archimedes had been persuaded by his friend and relation King Hieron to build such machines:-

These machines [*Archimedes*] *had designed and contrived, not as matters of any importance, but as mere amusements in geometry; in compliance with King Hiero's desire and request, some little time before, that he should reduce to practice some part of his admirable speculation in science, and by accommodating the theoretic truth to sensation and ordinary use, bring it more within the appreciation of the people in general.*

Perhaps it is sad that engines of war were appreciated by the people of this time in a way that theoretical mathematics was not, but one would have to remark that the world is not a very different place at the end of the second millenium AD. Other inventions of Archimedes such as the compound pulley also brought him great fame among his contemporaries. Again we quote Plutarch:-

[*Archimedes*] *had stated* [*in a letter to King Hieron*] *that given the force, any given weight might be moved, and even boasted, we are told, relying on the strength of demonstration, that if there were another earth, by going into it he could remove this. Hiero being struck with amazement at this, and entreating him to make good this problem by actual experiment, and show some great weight moved by a small engine, he fixed accordingly upon a ship of burden out of the king's arsenal, which could not be drawn out of the dock without great labour and many men; and, loading her with many passengers and a full freight, sitting himself the while far off, with no great endeavour, but only holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.*

Yet Archimedes, although he achieved fame by his mechanical inventions, believed that pure mathematics was the only worthy pursuit. Again Plutarch describes beautifully Archimedes attitude, yet we shall see later that Archimedes did in fact use some very practical methods to discover results from pure geometry:-

Archimedes possessed so high a spirit, so profound a soul, and such treasures of scientific knowledge, that though these inventions had now obtained him the renown of more than human sagacity, he yet would not deign to leave behind him any commentary or writing on such subjects; but, repudiating as sordid and ignoble the whole trade of engineering, and every sort of art that lends itself to mere use and profit, he placed his whole affection and ambition in those purer speculations where there can be no reference to the vulgar needs of life; studies, the superiority of which to all others is unquestioned, and in which the only doubt can be whether the beauty and grandeur of the subjects examined, of the precision and cogency of the methods and means of proof, most deserve our admiration.

His fascination with geometry is beautifully described by Plutarch:-

Oftimes Archimedes' servants got him against his will to the baths, to wash and anoint him, and yet being there, he would ever be drawing out of the geometrical figures, even in the very embers of the chimney. And while they were anointing of him with oils and sweet savours, with his fingers he drew lines upon his naked body, so far was he taken from himself, and brought into ecstasy or trance, with the delight he had in the study of geometry.

The achievements of Archimedes are quite outstanding. He is considered by most historians of mathematics as one of the greatest mathematicians of all time. He perfected a method of integration which allowed him to find areas, volumes and surface areas of many bodies. Chasles said that Archimedes' work on integration (see [7]):-

... gave birth to the calculus of the infinite conceived and brought to perfection by Kepler, Cavalieri, Fermat, Leibniz and Newton.

Archimedes was able to apply the method of exhaustion, which is the early form of integration, to obtain a whole range of important results and we mention some of these in the descriptions of his works below. Archimedes also gave an accurate approximation to π and showed that he could approximate square roots accurately. He invented a system for expressing large numbers. In mechanics Archimedes discovered fundamental theorems concerning the centre of gravity of plane figures and solids. His most famous theorem gives the weight of a body immersed in a liquid, called Archimedes' principle.

The works of Archimedes which have survived are as follows. *On plane equilibriums* (two books), *Quadrature of the parabola*, *On the sphere and cylinder* (two books), *On spirals*, *On conoids and spheroids*, *On floating bodies* (two books), *Measurement of a circle*, and *The Sandreckoner.* In the summer of 1906, J L Heiberg, professor of classical philology at the University of Copenhagen, discovered a 10^{th} century manuscript which included Archimedes' work *The method.* This provides a remarkable insight into how Archimedes discovered many of his results and we will discuss this below once we have given further details of what is in the surviving books.

The order in which Archimedes wrote his works is not known for certain. We have used the chronological order suggested by Heath in [7] in listing these works above, except for *The Method* which Heath has placed immediately before *On the sphere and cylinder.* The paper [47] looks at arguments for a different chronological order of Archimedes' works.

The treatise *On plane equilibriums* sets out the fundamental principles of mechanics, using the methods of geometry. Archimedes discovered fundamental theorems concerning the centre of gravity of plane figures and these are given in this work. In particular he finds, in book 1, the centre of gravity of a parallelogram, a triangle, and a trapezium. Book two is devoted entirely to finding the centre of gravity of a segment of a parabola. In the *Quadrature of the parabola* Archimedes finds the area of a segment of a parabola cut off by any chord.

In the first book of *On the sphere and cylinder* Archimedes shows that the surface of a sphere is four times that of a great circle, he finds the area of any segment of a sphere, he shows that the volume of a sphere is two-thirds the volume of a circumscribed cylinder, and that the surface of a sphere is two-thirds the surface of a circumscribed cylinder including its bases. A good discussion of how Archimedes may have been led to some of these results using infinitesimals is given in [14]. In the second book of this work Archimedes' most important result is to show how to cut a given sphere by a plane so that the ratio of the volumes of the two segments has a prescribed ratio.

In *On spirals* Archimedes defines a spiral, he gives fundamental properties connecting the length of the radius vector with the angles through which it has revolved. He gives results on tangents to the spiral as well as finding the area of portions of the spiral. In the work *On conoids and spheroids* Archimedes examines paraboloids of revolution, hyperboloids of revolution, and spheroids obtained by rotating an ellipse either about its major axis or about its minor axis. The main purpose of the work is to investigate the volume of segments of these three-dimensional figures. Some claim there is a lack of rigour in certain of the results of this work but the interesting discussion in [43] attributes this to a modern day reconstruction.

On floating bodies is a work in which Archimedes lays down the basic principles of hydrostatics. His most famous theorem which gives the weight of a body immersed in a liquid, called *Archimedes' principle,* is contained in this work. He also studied the stability of various floating bodies of different shapes and different specific gravities. In *Measurement of the Circle* Archimedes shows that the exact value of π lies between the values $3^{10}/_{71}$ and $3^{1}/_{7}$. This he obtained by circumscribing and inscribing a circle with regular polygons having 96 sides.

The Sandreckoner is a remarkable work in which Archimedes proposes a number system capable of expressing numbers up to 8×10^{63} in modern notation. He argues in this work that this number is large enough to count the number of grains of sand which could be fitted into the universe. There are also important historical remarks in this work, for Archimedes has to give the dimensions of the universe to be able to count the number of grains of sand which it could contain. He states that Aristarchus has proposed a system with the sun at the centre and the planets, including the Earth, revolving round it. In quoting results on the dimensions he states results due to Eudoxus, Phidias (his father), and to Aristarchus. There are other sources which mention Archimedes' work on distances to the heavenly bodies. For example in [59] Osborne reconstructs and discusses:-

...a theory of the distances of the heavenly bodies ascribed to Archimedes, but the corrupt state of the numerals in the sole surviving manuscript [*due to Hippolytus of Rome, about* 220 *AD*] *means that the material is difficult to handle.*

In the *Method*, Archimedes described the way in which he discovered many of his geometrical results (see [7]):-

... certain things first became clear to me by a mechanical method, although they had to be proved by geometry afterwards because their investigation by the said method did not furnish an actual proof. But it is of course easier, when we have previously acquired, by the method, some knowledge of the questions, to supply the proof than it is to find it without any previous knowledge.

Perhaps the brilliance of Archimedes' geometrical results is best summed up by Plutarch, who writes:-

It is not possible to find in all geometry more difficult and intricate questions, or more simple and lucid explanations. Some ascribe this to his natural genius; while others think that incredible effort and toil produced these, to all appearances, easy and unlaboured results. No amount of investigation of yours would succeed in attaining the proof, and yet, once seen, you immediately believe you would have discovered it; by so smooth and so rapid a path he leads you to the conclusion required.

Heath adds his opinion of the quality of Archimedes' work [7]:-

The treatises are, without exception, monuments of mathematical exposition; the gradual revelation of the plan of attack, the masterly ordering of the propositions, the stern elimination of everything not immediately relevant to the purpose, the finish of the whole, are so impressive in their perfection as to create a feeling akin to awe in the mind of the reader.

There are references to other works of Archimedes which are now lost. Pappus refers to a work by Archimedes on semi-regular polyhedra, Archimedes himself refers to a work on the number system which he proposed in the *Sandreckoner*, Pappus mentions a treatise *On balances and levers*, and Theon mentions a treatise by Archimedes about mirrors. Evidence for further lost works are discussed in [67] but the evidence is not totally convincing.

Archimedes was killed in 212 BC during the capture of Syracuse by the Romans in the Second Punic War after all his efforts to keep the Romans at bay with his machines of war had failed. Plutarch recounts three versions of the story of his killing which had come down to him. The first version:-

Archimedes ... was ..., as fate would have it, intent upon working out some problem by a diagram, and having fixed his mind alike and his eyes upon the subject of his speculation, he never noticed the incursion of the Romans, nor that the city was taken. In this transport of study and contemplation, a soldier, unexpectedly coming up to him, commanded him to follow to Marcellus; which he declining to do before he had worked out his problem to a demonstration, the soldier, enraged, drew his sword and ran him through.

The second version:-

... a Roman soldier, running upon him with a drawn sword, offered to kill him; and that Archimedes, looking back, earnestly besought him to hold his hand a little while, that he might not leave what he was then at work upon inconclusive and imperfect; but the soldier, nothing moved by his entreaty, instantly killed him.

Finally, the third version that Plutarch had heard:-

... as Archimedes was carrying to Marcellus mathematical instruments, dials, spheres, and angles, by which the magnitude of the sun might be measured to the sight, some soldiers seeing him, and thinking that he carried gold in a vessel, slew him.

Archimedes considered his most significant accomplishments were those concerning a cylinder circumscribing a sphere, and he asked for a representation of this together with his result on the ratio of the two, to be inscribed on his tomb. Cicero was in Sicily in 75 BC and he writes how he searched for Archimedes tomb (see for example [1]):-

... and found it enclosed all around and covered with brambles and thickets; for I remembered certain doggerel lines inscribed, as I had heard, upon his tomb, which stated that a sphere along with a cylinder had been put on top of his grave. Accordingly, after taking a good look all around ..., I noticed a small column arising a little above the bushes, on which there was a figure of a sphere and a cylinder... . Slaves were sent in with sickles ... and when a passage to the place was opened we approached the pedestal in front of us; the epigram was traceable with about half of the lines legible, as the latter portion was worn away.

It is perhaps surprising that the mathematical works of Archimedes were relatively little known immediately after his death. As Clagett writes in [1]:-

Unlike the Elements of Euclid, the works of Archimedes were not widely known in antiquity. ... It is true that ... individual works of Archimedes were obviously studied at Alexandria, since Archimedes was often quoted by three eminent mathematicians of Alexandria: Heron, Pappus and Theon.

Only after Eutocius brought out editions of some of Archimedes works, with commentaries, in the sixth century AD were the remarkable treatises to become more widely known. Finally, it is worth remarking that the test used today to determine how close to the original text the various versions of his treatises of Archimedes are, is to determine whether they have retained Archimedes' Dorian dialect.

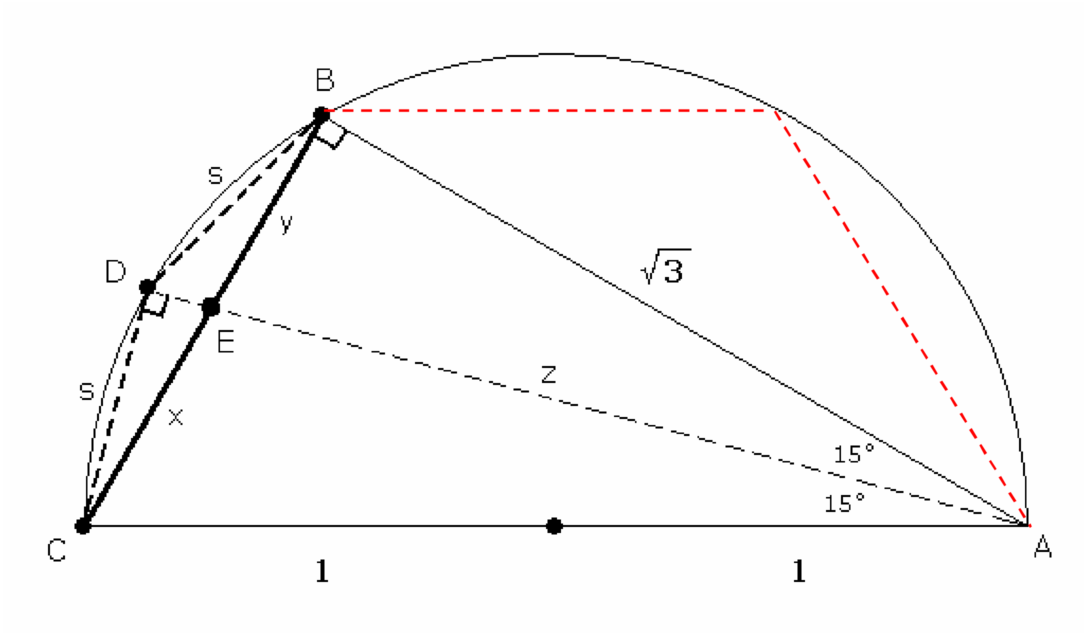
Article by: *J J O'Connor* and *E F Robertson*

January 1999

** See next page for p estimation theorem outline **

π AS A LIMIT : The Method of "Exhaustion"

(Archimedes, c. 225 BC - Greece)



Given: ABC is a unit semicircle ; BC is one side of inscribed regular hexagon
$\angle BAC = 30°$
AD bisects $\angle BAC$

True: ΔABC is a 30-60-90 right triangle
$AB = \sqrt{3}$
$BC = x + y = 1$
$CD = BD = s$
$AD = z = \sqrt{4 - s^2}$

by AA: ΔADC and ΔABE are right triangles and $\angle DAC = \angle BAE$,
$\Rightarrow \Delta DAC \sim \Delta BAE$

The'rm: $\frac{2}{x} = \frac{\sqrt{3}}{y} = \frac{2+\sqrt{3}}{x+y} = \frac{2+\sqrt{3}}{1} \quad \left(= \frac{1}{2-\sqrt{3}} \right)$

Pythag: $s^2 + z^2 = 4$
$\Rightarrow \quad z = \sqrt{4 - s^2}$

CPSTP: $\frac{\sqrt{3}}{y} = \frac{z}{s} = \frac{\sqrt{4-s^2}}{s} = \underline{2+\sqrt{3}}$

Thus: $\frac{4-s^2}{s^2} = \frac{4}{s^2} - 1 = \left(2+\sqrt{3}\right)^2 = 7 + 4\sqrt{3}$

$\Rightarrow \quad s^2 = \frac{1}{2+\sqrt{3}} = 2 - \sqrt{3} \quad ; \quad so, \quad s = \boxed{\sqrt{2-\sqrt{3}}}$

Definition: $\pi = \dfrac{circumference}{diameter} = \dfrac{1}{2} \cdot circumference = length\ of\ \ semicircle\ arc\ \ ABDC$

Inscribed Estimate:

#	n-gon	Exact $\left(\frac{n}{2} \cdot s\right)$	Decimal
1	6	$3 \cdot 1$	3.00000
2	12	$6 \cdot \sqrt{2-\sqrt{3}}$	3.10583
3	24	$12 \cdot \sqrt{2-\sqrt{2+\sqrt{3}}}$	3.13263
4	48	$24 \cdot \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$	3.13935
5	96	$48 \cdot \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$	<u>3.14103</u>

Circum-scribed Estimate:

#	n-gon	Exact $\left(\frac{n}{2} \cdot s\right)$	Decimal
1	6	$3 \cdot \dfrac{2}{\sqrt{3}}$	3.46410
2	12	$\dfrac{6\sqrt{2-\sqrt{3}} \cdot 2}{\sqrt{2+\sqrt{3}}}$	3.21539
3	24	$\dfrac{12\sqrt{2-\sqrt{2+\sqrt{3}}} \cdot 2}{\sqrt{2+\sqrt{2+\sqrt{3}}}}$	3.15966
4	48	$\dfrac{24\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}} \cdot 2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$	3.14609
5	96	$\dfrac{48\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}} \cdot 2}{\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}}$	<u>3.14271</u>

Mean: $3.140845 = 3\frac{10}{71} < 3.14103 < \pi < 3.14271 < 3\frac{1}{7} = 3.142857 \quad \left(\frac{265}{153} < \sqrt{3} < \frac{1351}{780}\right)$

Thus: $\pi \approx \dfrac{3\frac{10}{71} + 3\frac{1}{7}}{2} = \boxed{3.14185}$; $\%\,error = \dfrac{3.14185 - \pi}{\pi} \cdot 100 \approx .008 < \dfrac{1}{100}\,\%$