Euclid of Alexandria

Born: about 325 BC
Died: about 265 BC in Alexandria, Egypt

Euclid of Alexandria is the most prominent mathematician of antiquity best known for his treatise on mathematics *The Elements*. The long lasting nature of *The Elements* must make Euclid the leading mathematics teacher of all time. However little is known of Euclid's life except that he taught at Alexandria in Egypt. Proclus, the last major Greek philosopher, who lived around 450 AD wrote (see [1] or [9] or many other sources):-

Not much younger than these [*pupils of Plato*] *is Euclid, who put together the "Elements", arranging in order many of Eudoxus's theorems, perfecting many of Theaetetus's, and also bringing to irrefutable demonstration the things which had been only loosely proved by his predecessors. This man lived in the time of the first Ptolemy; for Archimedes, who followed closely upon the first Ptolemy makes mention of Euclid, and further they say that Ptolemy once asked him if there were a shorted way to study geometry than the Elements, to which he replied that there was no royal road to geometry. He is therefore younger than Plato's circle, but older than Eratosthenes and Archimedes; for these were contemporaries, as Eratosthenes somewhere says. In his aim he was a Platonist, being in sympathy with this philosophy, whence he made the end of the whole "Elements" the construction of the so-called Platonic figures.*

There is other information about Euclid given by certain authors but it is not thought to be reliable. Two different types of this extra information exists. The first type of extra information is that given by Arabian authors who state that Euclid was the son of Naucrates and that he was born in Tyre. It is believed by historians of mathematics that this is entirely fictitious and was merely invented by the authors.

The second type of information is that Euclid was born at Megara. This is due to an error on the part of the authors who first gave this information. In fact there was a Euclid of Megara, who was a philosopher who lived about 100 years before the mathematician Euclid of Alexandria. It is not quite the coincidence that it might seem that there were two learned men called Euclid. In fact Euclid was a very common name around this period and this is one further complication that makes it difficult to discover information concerning Euclid of Alexandria since there are references to numerous men called Euclid in the literature of this period.

Returning to the quotation from Proclus given above, the first point to make is that there is nothing inconsistent in the dating given. However, although we do not know for certain exactly what reference to Euclid in Archimedes' work Proclus is referring to, in what has come down to us there is only one reference to Euclid and this occurs in *On the sphere and the cylinder*. The obvious conclusion, therefore, is that all is well with the argument of Proclus and this was assumed until challenged by Hjelmslev in [48]. He argued that the reference to Euclid was added to Archimedes book at a later stage, and indeed it is a rather surprising reference. It was not the tradition of the time to give such references, moreover there are many other places in Archimedes where it would be appropriate to refer to Euclid and there is no such reference. Despite Hjelmslev's claims that the passage has been added later, Bulmer-Thomas writes in [1]:-

Although it is no longer possible to rely on this reference, a general consideration of Euclid's works ... still shows that he must have written after such pupils of Plato as Eudoxus and before Archimedes.

For further discussion on dating Euclid, see for example [8]. This is far from an end to the arguments about Euclid the mathematician. The situation is best summed up by Itard [11] who gives three possible hypotheses.

(i) Euclid was an historical character who wrote the *Elements* and the other works attributed to him.

(ii) Euclid was the leader of a team of mathematicians working at Alexandria. They all contributed to writing the 'complete works of Euclid', even continuing to write books under Euclid's name after his death.

(iii) Euclid was not an historical character. The 'complete works of Euclid' were written by a team of mathematicians at Alexandria who took the name Euclid from the historical character Euclid of Megara who had lived about 100 years earlier.

It is worth remarking that Itard, who accepts Hjelmslev's claims that the passage about Euclid was added to Archimedes, favours the second of the three possibilities that we listed above. We should, however, make some comments on the three possibilities which, it is fair to say, sum up pretty well all possible current theories.

There is some strong evidence to accept (i). It was accepted without question by everyone for over 2000 years and there is little evidence which is inconsistent with this hypothesis. It is true that there are differences in style between some of the books of the *Elements* yet many authors vary their style. Again the fact that Euclid undoubtedly based the *Elements* on previous works means that it would be rather remarkable if no trace of the style of the original author remained.

Even if we accept (i) then there is little doubt that Euclid built up a vigorous school of mathematics at Alexandria. He therefore would have had some able pupils who may have helped out in writing the books. However hypothesis (ii) goes much further than this and would suggest that different books were written by different mathematicians. Other than the differences in style referred to above, there is little direct evidence of this.

Although on the face of it (iii) might seem the most fanciful of the three suggestions, nevertheless the 20^{th} century example of Bourbaki shows that it is far from impossible. Henri Cartan, André Weil, Jean Dieudonné, Claude Chevalley, and Alexander Grothendieck wrote collectively under the name of Bourbaki and Bourbaki's *Eléments de mathématiques* contains more than 30 volumes. Of course if (iii) were the correct hypothesis then Apollonius, who studied with the pupils of Euclid in Alexandria, must have known there was no person 'Euclid' but the fact that he wrote:-

.... Euclid did not work out the syntheses of the locus with respect to three and four lines, but only a chance portion of it ...

certainly does not prove that Euclid was an historical character since there are many similar references to Bourbaki by mathematicians who knew perfectly well that Bourbaki was fictitious. Nevertheless the mathematicians who made up the Bourbaki team are all well known in their own right and this may be the greatest argument against hypothesis (iii) in that the 'Euclid team' would have to have consisted of outstanding mathematicians. So who were they?

We shall assume in this article that hypothesis (i) is true but, having no knowledge of Euclid, we must concentrate on his works after making a few comments on possible historical events. Euclid must have studied in Plato's Academy in Athens to have learnt of the geometry of Eudoxus and Theaetetus of which he was so familiar.

None of Euclid's works have a preface, at least none has come down to us so it is highly unlikely that any ever existed, so we cannot see any of his character, as we can of some other Greek mathematicians, from the nature of their prefaces. Pappus writes (see for example [1]) that Euclid was:-

... most fair and well disposed towards all who were able in any measure to advance mathematics, careful in no way to give offence, and although an exact scholar not vaunting himself.

Some claim these words have been added to Pappus, and certainly the point of the passage (in a continuation which we have not quoted) is to speak harshly (and almost certainly unfairly) of Apollonius. The picture of Euclid drawn by Pappus is, however, certainly in line with the evidence from his mathematical texts. Another story told by Stobaeus [9] is the following:-

... someone who had begun to learn geometry with Euclid, when he had learnt the first theorem, asked Euclid "What shall I get by learning these things?" Euclid called his slave and said "Give him threepence since he must make gain out of what he learns".

Euclid's most famous work is his treatise on mathematics *The Elements*. The book was a compilation of knowledge that became the centre of mathematical teaching for 2000 years. Probably no results in *The Elements* were first proved by Euclid but the organisation of the material and its exposition are certainly due to him. In fact there is ample evidence that Euclid is using earlier textbooks as he writes the *Elements* since he introduces quite a number of definitions which are never used such as that of an oblong, a rhombus, and a rhomboid.

The *Elements* begins with definitions and five postulates. The first three postulates are postulates of construction, for example the first postulate states that it is possible to draw a straight line between any two points. These postulates also implicitly assume the existence of points, lines and circles and then the existence of other geometric objects are deduced from the fact that these exist. There are other assumptions in the postulates which are not explicit. For example it is assumed that there is a unique line joining any two points. Similarly postulates two and three, on producing straight lines and drawing circles, respectively, assume the uniqueness of the objects the possibility of whose construction is being postulated.

The fourth and fifth postulates are of a different nature. Postulate four states that all right angles are equal. This may seem "obvious" but it actually assumes that space in homogeneous - by this we mean that a figure will be independent of the position in space in which it is placed. The famous fifth, or parallel, postulate states that one and only one line can be drawn through a point parallel to a given line. Euclid's decision to make this a postulate led to Euclidean geometry. It was not until the 19^{th} century that this postulate was dropped and non-euclidean geometries were studied.

There are also axioms which Euclid calls 'common notions'. These are not specific geometrical properties but rather general assumptions which allow mathematics to proceed as a deductive science. For example:-

Things which are equal to the same thing are equal to each other.

Zeno of Sidon, about 250 years after Euclid wrote the *Elements*, seems to have been the first to show that Euclid's propositions were not deduced from the postulates and axioms alone, and Euclid does make other subtle assumptions.

The *Elements* is divided into 13 books. Books one to six deal with plane geometry. In particular books one and two set out basic properties of triangles, parallels, parallelograms, rectangles and squares. Book three studies properties of the circle while book four deals with problems about circles and is thought largely to set out work of the followers of Pythagoras. Book five lays out the work of Eudoxus on proportion applied to commensurable and incommensurable magnitudes. Heath says [9]:-

Greek mathematics can boast no finer discovery than this theory, which put on a sound footing so much of geometry as depended on the use of proportion.

Book six looks at applications of the results of book five to plane geometry.

Books seven to nine deal with number theory. In particular book seven is a self-contained introduction to number theory and contains the Euclidean algorithm for finding the greatest common divisor of two numbers. Book eight looks at numbers in geometrical progression but van der Waerden writes in [2] that it contains:-

... cumbersome enunciations, needless repetitions, and even logical fallacies. Apparently Euclid's exposition excelled only in those parts in which he had excellent sources at his disposal.

Book ten deals with the theory of irrational numbers and is mainly the work of Theaetetus. Euclid changed the proofs of several theorems in this book so that they fitted the new definition of proportion given by Eudoxus.

Books eleven to thirteen deal with three-dimensional geometry. In book thirteen the basic definitions needed for the three books together are given. The theorems then follow a fairly similar pattern to the two-dimensional analogues previously given in books one and four. The main results of book twelve are that circles are to one another as the squares of their diameters and that spheres are to each other as the cubes of their diameters. These results are certainly due to Eudoxus. Euclid proves these theorems using the "method of exhaustion" as invented by Eudoxus. The *Elements* ends with book thirteen which discusses the properties of the five regular polyhedra and gives a proof that there are precisely five. This book appears to be based largely on an earlier treatise by Theaetetus.

Euclid's *Elements* is remarkable for the clarity with which the theorems are stated and proved. The standard of rigour was to become a goal for the inventors of the calculus centuries later. As Heath writes in [9]:-

This wonderful book, with all its imperfections, which are indeed slight enough when account is taken of the date it appeared, is and will doubtless remain the greatest mathematical textbook of all time. ... Even in Greek times the most accomplished mathematicians occupied themselves with it: Heron, Pappus, Porphyry, Proclus and Simplicius wrote commentaries; Theon of Alexandria re-edited it, altering the language here and there, mostly with a view to greater clearness and consistency...

It is a fascinating story how the *Elements* has survived from Euclid's time and this is told well by Fowler in [7]. He describes the earliest material relating to the *Elements* which has survived:-

Our earliest glimpse of Euclidean material will be the most remarkable for a thousand years, six fragmentary ostraca containing text and a figure ... found on Elephantine Island in 1906/07 *and* 1907/08*... These texts are early, though still more than* 100 *years after the death of Plato (they are dated on palaeographic grounds to the third quarter of the third century BC); advanced (they deal with the results found in the "Elements" [book thirteen] ... on the pentagon, hexagon, decagon, and icosahedron); and they do not follow the text of the Elements. ... So they give evidence of someone in the third century BC, located more than 500 miles south of Alexandria, working through this difficult material... this may be an attempt to understand the mathematics, and not a slavish copying ...*

The next fragment that we have dates from 75 - 125 AD and again appears to be notes by someone trying to understand the material of the Elements.

More than one thousand editions of The Elements have been published since it was first printed in 1482. Heath [9] discusses many of the editions and describes the likely changes to the text over the years.

B L van der Waerden assesses the importance of the Elements in [2]:-

Almost from the time of its writing and lasting almost to the present, the Elements has exerted a continuous and major influence on human affairs. It was the primary source of geometric reasoning, theorems, and methods at least until the advent of non-Euclidean geometry in the 19th century. It is sometimes said that, next to the Bible, the "Elements" may be the most translated, published, and studied of all the books produced in the Western world.

Euclid also wrote the following books which have survived: Data (with 94 propositions), which looks at what properties of figures can be deduced when other properties are given; On Divisions which looks at constructions to divide a figure into two parts with areas of given ratio; Optics which is the first Greek work on

perspective; and Phaenomena which is an elementary introduction to mathematical astronomy and gives results on the times stars in certain positions will rise and set. Euclid's following books have all been lost: Surface Loci (two books), Porisms (a three book work with, according to Pappus, 171 theorems and 38 lemmas), Conics (four books), Book of Fallacies and Elements of Music. The Book of Fallacies is described by Proclus [1]:-

Since many things seem to conform with the truth and to follow from scientific principles, but lead astray from the principles and deceive the more superficial, [Euclid] has handed down methods for the clear-sighted understanding of these matters also ... The treatise in which he gave this machinery to us is entitled Fallacies, enumerating in order the various kinds, exercising our intelligence in each case by theorems of all sorts, setting the true side by side with the false, and combining the refutation of the error with practical illustration.

Elements of Music is a work which is attributed to Euclid by Proclus. We have two treatises on music which have survived, and have by some authors attributed to Euclid, but it is now thought that they are not the work on music referred to by Proclus.

Euclid may not have been a first class mathematician but the long lasting nature of The Elements must make him the leading mathematics teacher of antiquity or perhaps of all time. As a final personal note let me add that my [EFR] own introduction to mathematics at school in the 1950s was from an edition of part of Euclid's Elements and the work provided a logical basis for mathematics and the concept of proof which seem to be lacking in school mathematics today.

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